Cauchy Condensation Test

Consider $\sum a_n$ $a_n \geq 0$
(a_n) decreasing

$\sum a_n$ converges $\Longleftrightarrow$
$\sum 2^n a_{2^n}$ converges.

Root Test:

If $\sum a_n$ is a series
and if $\lim |a_n|^{1/n}$ exists and
$= r$. Then

1. If $r < 1$, the series converges
2. If $r > 1$, the series diverges.

We are given $\lim |a_n|^{1/n} = r$.

1. Suppose $r < 1$. Let $\varepsilon = \frac{1-r}{2}$. Then $\exists N \in \mathbb{N}$

s.t $\forall n \geq N$, $\left| |a_n|^{1/n} - r \right| < \varepsilon$

$\Leftrightarrow r - \varepsilon < |a_n|^{1/n} < r + \varepsilon < 1$

$|A - B| < \varepsilon$
$\Leftrightarrow -\varepsilon < A - B < \varepsilon$
$B - \varepsilon < A < B + \varepsilon$

$\iff 0 \le |a_n| < (r+\varepsilon)^n$ pos. # < 1.

Since $\sum (r+\varepsilon)^n$ converges,
by comparison, $\sum_{n=N}^{\infty} |a_n|$ converges.

② $r > 1$ same sort of thing

$\varepsilon = \frac{r-1}{2}$.

$\underbrace{r-\varepsilon}_{\text{pos. \#} > 1} < |a_n|^{1/n}$

$\therefore (r-\varepsilon)^n < |a_n|$

... a_n does not converge to 0.

$\Rightarrow \sum a_n$ diverges.

2.22 Give example of altern. series where terms $\to 0$ but diverges.

$a_1 - a_2 + a_3 - a_4 + a_5 - \cdots$

$1 + \frac{1}{2}$

$1 + 1/2 + 1/3$

$1 - \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$

$$\underbrace{\underbrace{1 - \frac{1}{2} + 1}_{1+\frac{1}{2}} \underbrace{- \frac{1}{3} + \frac{2}{3}}_{+\frac{1}{3}}}$$

$$\underbrace{- \frac{1}{4} + \frac{2}{4} - +}$$

$S_{2n-1} = S_n^H$ ← n^{th} partial sum of $\sum \frac{1}{n}$. diverge

$S_1 = 1$

$n=2$ $S_3 = 1 + \frac{1}{2}$

$n=3$ $S_5 = 1 + \frac{1}{2} + \frac{1}{3}$

$\therefore$ (S_j) diverges.

3.2 C = Cantor set: I

find ⓐ I open interval s.t. $I \cap C$ closed

ⓑ J open - - - $I \cap C$ not closed.

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0 is a limit pt of C, (because $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \dots \to 0$.)

0 is a limit pt of $C \setminus \{0\}$.

But not included in $J \cap C = C \setminus \{0\}$.

$J = (0,5)$

3.12 $\forall\, x, y \in \mathbb{Q}$, $\exists$ separation $\{A, B\}$ of $\mathbb{Q}$

s.t. $\mathbb{Q} = A \sqcup B$, A, B nonempty, open $\cap\, \mathbb{Q}$

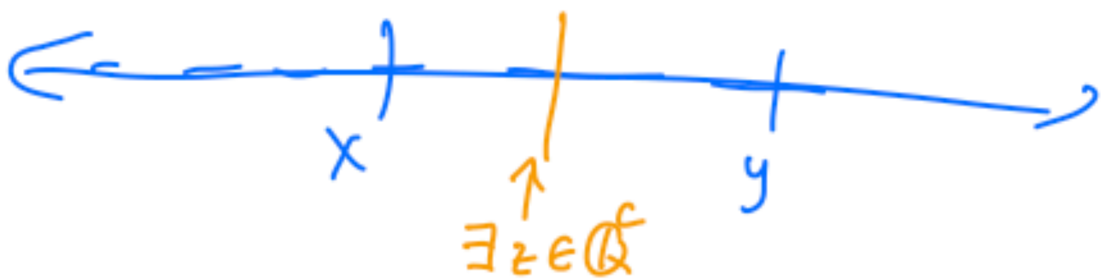
where $x \in A$, $y \in B$.

$A = \mathbb{Q} \cap U$

$B = \mathbb{Q} \cap V$

U, V open $U \cap V = \emptyset$.

$\mathbb{Q}$



$U = (-\infty, z)$, $V = (z, \infty)$

$$\mathbb{Q} = \underbrace{\mathbb{Q} \cap U}_{A} \cup \underbrace{\mathbb{Q} \cap V}_{B}.$$

3.9 $A \subseteq B$ closed. $\leftarrow L_B \subseteq B$

Then $\overline{A} \subseteq B$.

By defn. $\overline{A} = A \cup L_A$

Definitely $A \subseteq B$. (Given).

We just need to show

$L_A \subseteq B$.

Since every limit pt x of A,

$\exists$ seq. (x_n) in A s.t. $\lim x_n = x$,
(s.t. $x_n \neq x \ \forall n$).

Since $x_n \in A \subseteq B$ so each $x_n \in B$.

This x is a limit of B.

Since $\underline{B \text{ is closed}}$,

$x \in B$.

Note: If B is not closed, $A \subseteq B$, it may be the case that $\bar{A} \not\subseteq B$.

eg $A = (0,1)$

$B = (-1,1)$

$\bar{A} = [0,1]$,

$\bar{A} \not\subseteq B$.

3.7 Give an example of a closed set A where there is a countably infinite subset S does not have a limit pt in A.

To make example: S should not have any limit point.

$\Rightarrow$ unbounded (converse of BW)

Let $A = \mathbb{R}$

$B = \mathbb{N}$. ← No limit pts — (all pts are isolated.)

$A = [0, \infty)$

$\boxed{3.6}$ Give an example where $A \subseteq \mathbb{R}$, $L_A = L_{L_A}$.

Boring example $A = \mathbb{R}$ ✓
$L_A = \mathbb{R}$

$(x + \frac{1}{n}) \to x$

✓ $A = \mathbb{Q}$, $L_A = \mathbb{R}$

$L_{L_A} = L_{\mathbb{R}} = \mathbb{R}$.

$A = \mathbb{N}$, $L_A = \emptyset$, $L_\emptyset = \emptyset$.

$A = [0,1] \Rightarrow L_A = [0,1]$,

$L_{L_A} = [0,1]$.

$A = (0,1) \cap \mathbb{Q}$

$L_A = [0,1]$

$L_{L_A} = [0,1]$.

Example where $L_A \neq L_{L_A}$:

$A = \{\frac{1}{n} : n \in \mathbb{N}\}$.

$L_A = \{0\}$. $L_{L_A} = L_{\{0\}} = \emptyset$.

<u>Thm</u> $L_{L_A} \subset L_A$. <u>Cor.</u> $\bar{A}$ is closed.

Find $\bar{\bar{\bar{A}}}$

Ans. $\bar{A}$.

3.1 $[0,1)\cup(1,2] = S$

(a) show not closed

(b) show not open

(a) We'll show 1 is a limit pt of S.

$\forall \varepsilon' > 0$, let $\varepsilon = \min\{\varepsilon', \frac{1}{2}\}$.

$$V_{\varepsilon'}(1) \supseteq V_{\varepsilon}(1) = (1-\varepsilon, 1+\varepsilon)$$

contains $1+\frac{\varepsilon}{2} \in S$,

$1+\frac{\varepsilon}{2} \neq 1$.

$\therefore$ 1 is a limit pt of S.

(b) $\exists 0 \in S$.

$\forall \varepsilon > 0$, $V_{\varepsilon}(0) \not\subseteq S$.

$(-\varepsilon, \varepsilon)$ because

$-\frac{\varepsilon}{2}$